MATH2050C Assignment 9

Deadline: March 21 , 2018.

Hand in: 4.3. no. 5ab, 11; Supp. Ex. 2, 4.

Section 4.3 no. 3, 4, 5abedh, 8, 11.

Supplementary Exercises

Justify your answers in the following problems.

1. Evaluate
$$\lim_{x\to 2}\frac{x^2-4}{x-2}.$$

2. Evaluate
$$\lim_{x\to -3}\frac{x^2-2x-15}{x+3}.$$

3. Evaluate
$$\lim_{x\to \infty}\frac{\cos x}{x}.$$

4. Evaluate
$$\lim_{x\to x_0}\frac{\sin x-\sin x_0}{x-x_0}\ .$$
Hint: Let $h=x-x_0$ and reduce the problem to $h\to 0$. Then make use of the compound angle formula for the sine function.

Further Comments on Limits of Functions

First, we have studied limits of functions. For polynomials and rational functions, their limits are well understood. Indeed, let $r(x) = p(x)/q(x)$ be a rational function. We knew (1) it is well defined on the set $E = \{x \in \mathbb{R} : \ q(x) \neq 0\}$, (since a polynomial has at most finitely many roots, E is the union of finitely many open intervals.) (2) $\lim_{x\to x_0} r(x) = r(x_0)$ whenever x_0 satisfies $q(x_0) \neq 0$.

In order to have more examples to work on, we need to introduce more functions. In this chapter the following functions are studied:

- The **square root** $f_1(x) = \sqrt{x}$. It is defined on $[0, \infty)$ and $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$ for all $x_0 \geq 0$. See Ex 8 for a more general result.
- The **(rational) power** $f_2(x) = x^{m/n}$. Generalizing the square root, it is known from the last chapter that for each $x \geq 0$, there is a unique $y \geq 0$ satisfying $y^n = x$. We write $y = x^{1/n}$ the n-th root of x. Then $x^{m/n} = (x^{1/n})^m, x \in [0, \infty)$, is well-defined for all $n, m \in \mathbb{N}$. We also define $x^{-m/n} = 1/x^{m/n}$.
- The **absolute value function** $f_3(x) = |f(x)|$. It is defined on $(-\infty, \infty)$ and $\lim_{x\to x_0} |x| = |x_0|$ for all $x_0 \in (-\infty, \infty)$. See Ex 8 for a more general result.
- The **sine function** $f_4(x) = \sin x, x \in \mathbb{R}$. We do not need a rigorous definition here. Simply assuming that it is an odd function satisfying $x - x^3/6 \leq \sin x \leq x$ for $x \geq 0$, we deduce $\lim_{x\to 0} \sin x/x = 1$.
- The **cosine function** $f_5(x) = \cos x, x \in \mathbb{R}$. Again we do not need a rigorous definition. Simply assuming that it is an even function satisfying $1 - x^2/2 \leq \cos x \leq 1$ for all $x \geq 0$, we deduce $\lim_{x\to 0}(\cos x - 1)/x = 0$.
- The **exponential function** $f_6(x) = e^x, x \in \mathbb{R}$. When $x \geq 0$, we proved in Ex 5 that $e^x = \lim_{n\to\infty}(1 + x/n)^n = \sum_{n=0}^{\infty} x^n/n!$ and define $e^x = 1/e^{-x}$ for $x < 0$. Assuming that $e^x \geq x$ for $x \geq 0$, we can prove $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$. Also $\lim_{x\to 0^+} e^{1/x} = \infty$ and $\lim_{x\to 0^-} e^{1/x} = 0$.

In the next chapter, we will show that positive powers, sine, cosine and exponential functions all satisfy $\lim_{x\to x_0} f(x) = f(x_0)$ for all $x_0 \in \mathbb{R}$, that is, they are continuous everywhere.

Second, variatons on the notion of limits of functions including divergence at infinity and limits at infinity. Let f be function defined on $(a, b]$. It is said to **tend to** ∞ **(resp.** $-\infty$) at a if for each $M > 0$, there is some $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) for all $x \in (a, a + \delta)$. The notation is $\lim_{x\to a^+} f(x) = \infty$ (resp. $\lim_{x\to a^+} f(x) = -\infty$). Similarly, one can define $\lim_{x\to b^-} f(x) = \pm\infty$. For f defined on (a, ∞) (resp. $(-\infty, b)$) we can define $\lim_{x\to\infty} f(x) = L$ if for each $\varepsilon > 0$ there is some $K > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > K$. Similarly, we can define $\lim_{x\to-\infty} f(x) = L$, $\lim_{x\to\infty} f(x) = \pm\infty$, $\lim_{x\to-\infty} f(x) = \pm\infty$, etc. For these variations of limits of functions, the corresponding Sequential Criterion, Limit Theorems, and Squeeze Theorem are for you to explore, or simply look up the text book.